QUALITATIVE STUDY OF EQUATIONS DESCRIBING QUASI-ONE-DIMENSIONAL NONEQUILIBRIUM DUCT FLOW

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A qualitative study of the system of differential equations for the quasi-one-dimensional nonequilibrium steady-state duct flow allows one to describe the possible types of flow depending on the geometry of the duct and kinetic equations governing the relaxation process. The advantage of such a treatment lies in the opportunity to use the results of this analysis in the numerical solution of problems on nonequilibrium fluid flow. The study is complicated by the fact that the physical model is described by incomplete system of equations. The method of analysis is given in [1, 2] for the phase portrait of such a system. This method is used here for the solution of the problem on the flow of a mixture of gas and liquid particles, taking into account the nonequilibrium crystallization in variable cross-sectional ducts.

1. Physical Formulation of the Problem. Reduction to the Normal Form. Derivation of the Characteristic Equation

Equations of quasi-one-dimensional flow with one nonequilibrium process in a variable area of cross-sectional duct have the form

$$uy = C_1 v,\ C_1 = u_0 y_0 / v_0,\ udu + vdp = 0,\ TdS = -e_\xi d\xi,\quad d\xi = (\varkappa/u)dx. \tag{1.1}$$

System (1.1) is supplemented by equation of state

$$p = -e_v(S, v, \xi),\ T = e_S(S, v, \xi), \tag{1.2}$$

where $\rho = 1/v$ is the density; v, specific volume; u, velocity; p, pressure; T, temperature; S, entropy; e, internal energy of the gas; ξ, relaxation parameter; $\varkappa$, source function for ξ; $y = y(x)$, equation describing the nozzle contour. M^2, v, S, and ξ (M^2 is the "frozen" Mach number) are chosen as the unknown functions and the Eqs. (1.1) and (1.2) are written in the form

$$\frac{\dot{M}^2}{M^2} = \frac{af - bc + d\Delta M^2}{\Delta M^2} \equiv \frac{P}{\Delta M^2},\ \frac{\dot{v}}{v} = \frac{\dot{M}^2}{f} - \frac{2b}{f} - \frac{M^2}{f}d,\quad \dot{S} = -\frac{e_\xi}{e_S}\dot{\xi},\quad \dot{\xi} = \frac{\varkappa}{u}, \tag{1.3}$$

where $a = \dot{y}/y$; $b = (e, p)\dot{\xi}/(ve_S p_c)$; for any φ, ψ let $(\varphi, \psi) = \partial(\varphi, \psi)/\partial(S, \xi)$; $c = vp_{VV}/p_V$; $d = -(e, p_V)\dot{\xi}/(e_S p_V p_V)$; $f = -2\Delta M^2 - cM^2$; $\Delta M^2 = M^2 - 1$; the dot denotes differentiation with respect to x.

In order to have a continuous solution for the system (1.3) having the value $M^2 = 1$, it is necessary to satisfy the condition that the denominator and the numerator go to zero simultaneously in the first equation in (1.3). It is possible to observe that the remaining equations will also have finite derivatives near the singularity. Thus, the nature of singularities in (1.3) as $M^2 = 1$ can be analyzed (as in [1]) on the basis of the method used in the study of the plane problem. The equation of the singular surface in the region of variation of the variables M^2, v, S, x, and ξ has the form $a + b = \dot{y}/y + (e, p)\dot{\xi}/ve_S p_V = 0$, $M^2 - 1 = 0$, and the equation for the determination of characteristic directions will be

$$K^2 - AK + B = 0, \tag{1.4}$$

where

$$A = \frac{\partial R}{\partial M^2} - \frac{v}{c}\frac{\partial P}{\partial v};\quad -B = \frac{\partial P}{\partial v}\frac{(2b + d)v}{c} + \frac{\partial P}{\partial S}\dot{S} + \frac{\partial P}{\partial \xi}\dot{\xi} + \frac{\partial P}{\partial x}$$

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(here the variables are taken on the characteristic surface).

The nature of singularities is determined from the sign of the roots of the equation $K_{1,2} = (A \pm \sqrt{A^2 - 4B})/2$, where the coefficients A, B after certain transformations take the form

$$A = -bv\frac{\partial}{\partial v}\ln\left[-\frac{\Delta\xi}{e_S}\right], \quad \Delta\xi = \xi - \xi_e,$$

$$B = b^2v^2\left[\frac{p_{vv}}{(e,\ p)}(\ln[-b],\ e) + \frac{\partial}{\partial v}\left[\ln\frac{v^2}{(e,\ p)}\right]\frac{\partial}{\partial v}\ln[-b]\right] + \dot{a}c.$$

The nature of singularities depend on the signs of K_i. When $B < 0$ the singularity is like a saddle point; when $B > 0$ and $A^2 - 4B > 0$ it is a multiple singularity ($A < 0$ has negative characteristic lines and $A > 0$ has positive characteristic lines); when $A^2 - 4B < 0$, it is a focal singularity [1]. Consider the following problem as an example.

2. Study of the Flow of a Mixture of Gas and Liquid (Solid) Particles Taking into Account Their Crystallization (Fusion) in Variable Cross-Sectional Ducts

Equations describing the above flow in the case of a difference in temperatures of the dispersed and continuous phases have the form [3]

$$\rho uy = C_1,\ e = c_V T + c_2 T_2 - L\xi, \tag{2.1}$$
$$udu + vdp = 0,\ p = \bar{\alpha}T/W,\ W = v - \beta - \delta\xi,$$
$$e + pv + u^2/2 = C_3,\ \rho_{22} = r,\ \rho_{33} = s,\ r,\ s = \text{const},$$
$$ud\xi = -(1/\tau)\Delta\xi dx,\ \xi_e = \xi_0\exp(\zeta - \zeta_0),\ \zeta = (L - \delta p)/T,$$
$$udT_2 = -(1/\tau_1)(T_2 - T)dx,$$

where ρ, u, p, e are known quantities for the mixture; $\rho = \sum_1^3 \rho_i$, $\rho_i = m_i\rho_{ii}$, average density of the phase (i = 1 is for gas, i = 2 is for liquid, and i = 3 is for the solid component of the mixture); m_i, volumetric concentration; ρ_{ii}, true density; T, T_2, temperatures of the continuous and dispersed phases; $\rho_i = \bar{\alpha}\rho$; $\rho_2 = (\alpha - \xi)\rho$; $c_V = \bar{\alpha}c_{v1}$; $\bar{\alpha} = 1 - \alpha$; $c_2 = \alpha c_2^*$; $\alpha = (\rho_{2,0} + \rho_{3,0})/\rho_0$; $\beta = \alpha/r$; $\delta = 1/r - 1/s$; τ, τ_1, relaxation times for the parameters ξ, T_2; c_{V1}, c_2^*, specific heats of the phases; $\xi = \rho_3 v$, relative mass concentration of the solid phase; L, latent heat of phase transition. The index zero refers to a certain initial conditions at which C_1 and C_3 are determined.

It is possible to determine the following limiting variants of the model (5).

1. If $\tau_1 \to \infty$, $\tau \to \infty$, then $T_2 \to$ const, $\xi \to$ const. This condition corresponds to the flow with frozen crystallization when the gasdynamic parameters of the continuous phase vary much more rapidly than the thermodynamic parameter (T_2) of the dispersion phase. Denote it by the variant $\Gamma(\infty, \infty)$, in [4] it corresponds to instantaneous freezing.

2. If $\tau_1 \to 0$, $\tau \neq 0, \infty$, then this variant is denoted by $\Gamma(\tau, 0)$, it corresponds to instantaneous effect of the temperature of the dispersion phase on the change in the gaseous phase $T_2 = T$ [3].

3. If $\tau_1 \neq 0, \infty$, $\tau \to 0$, then the crystallization process is an equilibrium process in terms of the concentration, i.e., $\xi = \xi_e(T, T_2, p)$, $T \neq T_2$, variant $\Gamma(0, \tau_1)$.

The results of the qualitative analysis given in Sec. 1 are used for the relaxation process to describe the flow of the mixture of the type $\Gamma(\tau, 0)$. We shall also consider, for simplicity, the volumetric concentration of the dispersion phase to be small, i.e., $\beta \ll v$, and neglect the work expended on the change in volume during crystallization, i.e., assume $\delta \equiv 0$.

In this case the discriminant that determines the nature of singularities has the form

$$A^2 - 4B = D(\chi) = b^2(\chi^2 + A_1\chi + A_2),$$

where

$$A_1 = 4\gamma\left(n + h\frac{L_1 - \dot{\varphi}}{L_1}\right); \quad A_2 = 4\gamma(\gamma - 1)\left[-\frac{n}{2} + h\frac{L_1 - \dot{\varphi}}{L_1} - \right.$$

$$-h\frac{1{,}5L_1-1/\Delta\xi}{L_1}-h\frac{\gamma\dot{a}T}{L_1^2(\Delta\xi/\tau)^2}\Bigg];\quad \chi=\frac{v\xi_{e,v}}{\Delta\xi}-(\gamma-1);\quad L_1=\frac{L}{c_V T};$$

$$n=\frac{\gamma+2}{\gamma};\quad h=\frac{1+\gamma}{1-\gamma};\quad \dot{\varphi}=\frac{1}{c_V}\ln\left[\frac{\xi}{\xi_0}e^{-\zeta_0}\right].$$

For small latent heats for phase transformation L, the quantities χ and $(L_1-\dot{\varphi})/L_1$ are finite, the quantity $1/\Delta\xi \sim 0\,(L)$, and the sign of $D(\chi)$ is determined by the main term in A_2 of the type $A_3\dot{a}/L^4$. If we assume that the transition through the singular point takes place in the neighborhood of the point $x = x_*$ such that $\dot{y}(x_*) = 0$, then for $a(x)$ we have $a \simeq mk(k-1)\Delta x^{k-2}/y_*$, where $\Delta x = x - x_*$, $y_* = y(x_*)$. Then as $x > x_*$, $a > 0$, taking into account $A_3 > 0$, $A/b = \chi \sim 0(1)$, it is possible to formulate the following.

Theorem 1. In the flow of a mixture of gas and crystallizing particles in a Laval nozzle, equations describing the motion of the type $\Gamma(\tau, 0)$ for small values of latent heat of phase transformation, have a saddle-point type of singularity at the sonic point.

Consider the flow of the type $\Gamma(\infty, \tau_1)$ as the second example. In this case a certain simplification is possible for the general approach described in Sec. 1. Choosing M^2 and u as the unknown functions and having the relation of the type of the second equation from (1.3) for the derivatives of the unknown functions, it is possible to realize that the nature of the singularities at $M^2 = 1$ can be studied, in a manner similar to Sec. 1, on the basis of the method used for studying plane flows. The equation for the zeroth critical line in this case has the form

$$M^2-1=0,\quad a+c_2\bar{\gamma}(T_2-T)/u^2\tau_1=0$$

or

$$\tau_1 a u^3 + B_1 u^2 + D_1 = 0. \tag{2.2}$$

Here $B_1 = [(1+\gamma)\gamma - 2\gamma c_2]/2\gamma > 0$; $\bar{\gamma} = 1-\gamma$; $D_1 = C_3\bar{\gamma} < 0$. Consider the behavior of the roots $u = u(x)$ of this equation. Elementary analysis showed that as $a \in [-a_{\rm I}, a_{\rm I}] = J$, $a_{\rm I} = -(2D_1/\tau_1)\sqrt{-(B_1/3D_1)^3}$ the equation for zeroth critical line has three real roots; outside this interval, i.e., as $a \in J$ there is one real and two imaginary roots. If $a < 0$ and lies in the region J, there are two positive and negative roots, if, however, $a < 0$ and lies in R/J, then the only real root is negative. The consideration of $a > 0$ and $a \in J$ shows that in this case there are two negative and one positive root; $a > 0$, $a \in R/J$ has two complex and one real positive root.

In the neighborhood of the minimum cross-sectional point of the nozzle it is possible to write the solution in an explicit form with an accuracy up to $o(a^2)$:

$$u_{1,2}(x)=\pm\sqrt{-\frac{D_1}{B_1}+a\frac{\tau_1 D_1}{2B_1^2}+\ldots},\quad u_3(x)=-\frac{B_1}{\tau_1 a}+\ldots, \tag{2.3}$$

i.e., in the neighborhood of the minimum cross section of the nozzle $x = x_*$, the solution $u_{1,2}(x)$ remains continuous, $u_3(x)$ has a second-order discontinuity. Unlike the equilibrium and frozen flows of the gas in the Laval nozzle, the passage through the sonic speed in this case is achieved not when $x = x_*$, but, as it is well known, when there is heat release below this flow. Using the expansions $u = u_f + \tau_1^{-1}u_1 + \ldots$, $x = x_* + \tau_1^{-(k-1)}x_1 + \ldots$, , and the already known asymptote of the nozzle contour $(k = 2n)$ we find from (2.2) the first term of the expansion of the coordinate at the critical section

$$x_{\text{кр}}-x_*=\left(-\frac{B_1u_f^2+D_1}{ka_1u_f^2\tau_1}\right)^{\frac{1}{k-1}}+\ldots\equiv\left[\frac{c_2(\gamma-1)}{a_1ku_f^3\tau_1}(T_{2,0}-T_f)\right]^{\frac{1}{k-1}}+\ldots,$$

where

$$a_1=km/y_*;\quad u_f^2=\frac{2(c_3-c_2T_{2,0})(\gamma-1)}{\gamma+1};\quad T_f=\frac{2(c_3-c_2T_{2,0})(\gamma-1)}{\gamma(\gamma+1)}.$$

Since $T_{2.0} > T_f$ it follows that the critical in the nonequilibrium (near the frozen) flow is found downstream of x_*. After the determination of x_{cr} from Eq. (2.3) we find the unique positive solution to the Eq. (2.2). Thus we showed that there is a solution for the flow of the type $\Gamma(\infty, \tau_1)$ which transforms the denominator and the numerator to zero. Further analysis, similar to that described in Sec. 1, made it possible to formulate the following.

Theorem 2. For the flow of the mixture of gas and crystallizing particles in Laval nozzle, equations describing the flow of the type $\Gamma(\infty, \tau_1)$ have multiple singularity at the sonic point when

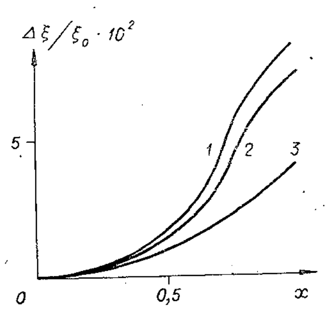

Fig. 1

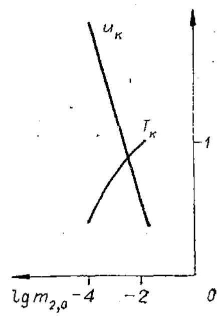

Fig. 2

$$U > U_4,\ t \in [0, t_3],\ U \in [U_5, U_3],\ t \in [0, t_3],$$
$$U \in (U_2, U_4),\ t \in (0, 1),\ U \in [U_1, U_2],\ t \in (0, 1),$$
$$U \in [U_5, U_1],\ t \in (0, t_1);\ U < U_5,\ t \in [t_2, t_3],$$

where

$$a_2 = \gamma(1 + \gamma - \bar{\gamma}c_2);\quad a_3 = a_2 + d_1 U;\quad b_1 = -\gamma(1 + \gamma) + 2\bar{\gamma}c_2;\quad c_1 = \bar{\gamma}^3 c_2/4;\quad d_1 = (1 - \gamma^2)c_2;\quad U = (\ddot{y}/y)/(\dot{y}/y)^2;\quad U_1 = (\gamma - 2)/(1 + \gamma);\quad U_2 = -a_2/d_1;$$
$$U_3 = U_2 + b_1^2/4d_1c_1;\quad U_4 = U_2 + b_1/d_1;\quad U_5 = U_2 - 2b_1^2/d_1c_1;$$
$$t_{1,2} = \left(-b_1 \pm \sqrt{b_1^2 - 4a_3c_1}\right)/2a_1;\quad t_3 = b_1/a_3;\quad t = 1 - T/T_2;$$

a singularity of the saddle point type if

$$U > U_3,\ t < 0,\ U \in (U_2, U_3),\ t < t_2,\ t \in (t_1, 0),\ U > U_5,\ t \in (t_3, 1),$$
$$U \in (U_5, U_2),\ t \in (t_2, 0),\ U < U_5,\ t \in (t_3, t_1);$$

and a singularity of the focal point type if

$$U \in (U_2, U_3),\ t \in (t_2, t_1),\ U \in [U_5, U_2],\ t < t_2,$$
$$U < U_1,\ t \in [t_1, 1].$$

Note 1. The existence of multiple singularity and positive roots are not possible.

3. An Example of Numerical Computations

The qualitative analysis of the system of equations describing quasi-one-dimensional nonequilibrium flow in nozzles showed that in the case $\Gamma(\tau, 0)$ the singularity at small L is a saddle point. This enables us to seek a solution in which the value of gas velocity passes through sonic speed, with the help of the selection of certain initial parameter, e.g., u_0. Numerically solving the Cauchy problem we move up to the point in the nozzle where $M = 1 - \varepsilon_1$. Then the difference $K_1 > 0$ (derivative to the solution near the singularity) computed from the equation given in Sec. 1 is checked with $K_1 > 0$ obtained numerically. If the difference is appreciable then u_0 is varied and the computations are carried out again. If the agreement is satisfactory, an extrapolation is made up to $M = 1 + \varepsilon_2$ and further computations are carried out till the end of the nozzle.

Computations were made according to the above algorithm. A weighted scheme was used in the solution procedure. The resulting nonlinear algebraic equations were solved using Newton's iterative procedure.

Nondimensionalizing was carried out in the standard manner, the reference quantities were: T_0, p_0 for temperature (equal to the fusion temperature) and gas pressure at the end of the volume ahead of the nozzle; $u_0^2 = RT_0$, R for specific heats of the phases; RT_0 for the latent heat of phase transformation. The equation describing equilibrium crystallization was taken in the form $\xi_e = \xi_0 \exp[K_3(\zeta - \zeta_0)]$.

The effect of relaxation time on the nature of the flow was studied during the numerical computations. When $\tau \to 0$ the value of the parameters of the mixture appeared to increase at fixed points of the nozzle, as shown in Fig. 1 for $\Delta\xi/\xi_0$ for different initial concentration (τ = 0.01, 0.1, 0.5, as indicated by lines 1-3). It is also seen here that with an increase in volumetric concentration of the liquid phase $m_{2,0}\Delta\xi$ varies more weakly. This is associated with the freezing of the expansion process at large concentrations of liquid particles which in its turn is caused by large heat capacity of the particles which maintain high temperature since they do not have time to cool during the motion through the nozzle.

The natural phenomenon of the reduction in flow velocity in the duct with increase in $M_{2.0}$ caused by energy expenditure on the particle transfer has also been observed. Here the introduction of heat on account of phase transformation cannot compensate these losses as shown in Fig. 2 in the form of a relation between velocity, temperature at the nozzle section u_k, T_k, and log $m_{2.0}$.

The effect of the latent heat of phase transformation L on the distribution of flow parameters along the nozzle axis has been studied. With an increase in the latent heat of phase transformation there was an increase in u, T caused by the introduction of additional heat. The increase in L by an order of magnitude leads to a change in u, p, T by an order.

Computations, described above, were carried out for $K_3 = 1$, $L = 0.34$ for Al_2O_3.

In order to verify the correctness of the computational program, the variant with $m_{2.0} = m_{3.0} = 10^{-10}$, $\tau = 10^{10}$ was computed and compared with analytical solution for the flow of an ideal gas in a Laval nozzle. The agreement between numerical and analytical solutions was good to an accuracy of up to a hundredth of a percent. The accuracy was also checked for the equilibrium relations, indicating sufficient accuracy of the computations.

Thus a qualitative analysis has been carried out for the system of equations describing nonequilibrium flow of a mixture. A numerical algorithm has been compiled on the basis of this analysis which made it possible to compute flow parameters of condensed products of combustion in a Laval nozzle, taking crystallization into account.

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